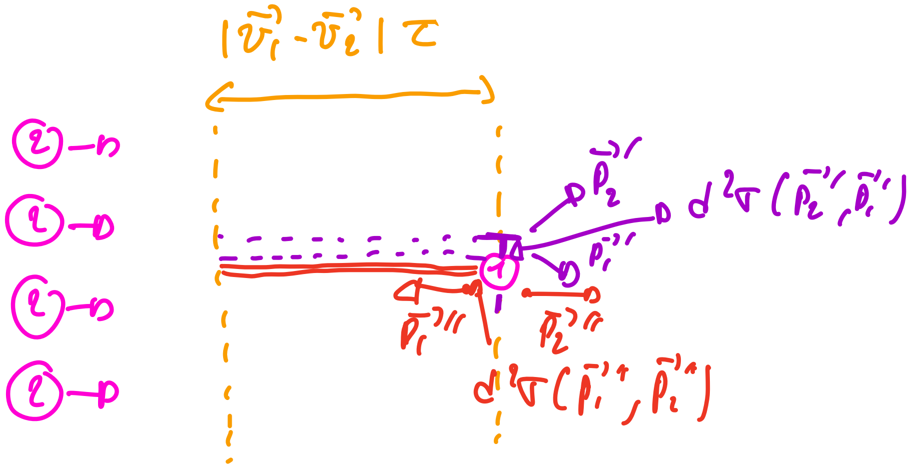
1

where $R^-(\theta,\varphi)d\Omega$ is the rate of collisions rotating $\vec{p}_1$ within a solid angle $d\Omega$ of $\mathrm{Rot}(\theta,\varphi)\cdot\vec{e}_1$

Cross section:

$$d^2\sigma(\theta,\varphi) = \frac{R^-(\theta,\varphi)\,d\Omega}{|v_1-v_2|\,\hat{f}_1(\vec{q}_1,\vec{p}_2)\,d^3\vec{p}_2} = \frac{\text{Rate of collision in given direction}}{\text{Incoming flux}}$$

$d^2\sigma(\theta,\varphi) = d^2\sigma(\vec{p}_1{}',\vec{p}_2{}')$ is an effective area that measures how efficient the collision is at producing $\vec{p}_1{}',\vec{p}_2{}'$ from $\vec{p}_1$ & $\vec{p}_2$.



All in all:

$$N^- = \int d^3\vec{p}_2 \int d^2\sigma(\theta,\varphi)\,\hat{f}_1(\vec{p}_1,\vec{q}_1,t)\,V_c\,d^3\vec{p}_1\,\hat{f}_1(\vec{p}_2,\vec{q}_1,t)\,|\vec{v}_1-\vec{v}_2|\,\tau$$

Now we can compute N^+!

$$dN^- \equiv dN(\vec{p}_2,\vec{p}_1 \to \vec{p}_2{}',\vec{p}_1{}')$$
$$= d^3\vec{p}_2\,d^3\vec{p}_1\,d^2\sigma(\theta,\varphi)\,V_c\,\hat{f}_1(\vec{p}_1,\vec{q}_1)\,\hat{f}_1(\vec{p}_2,\vec{q}_1)\,|\vec{v}_1-\vec{v}_2|\,\tau$$

Time-reversal & parity symmetry

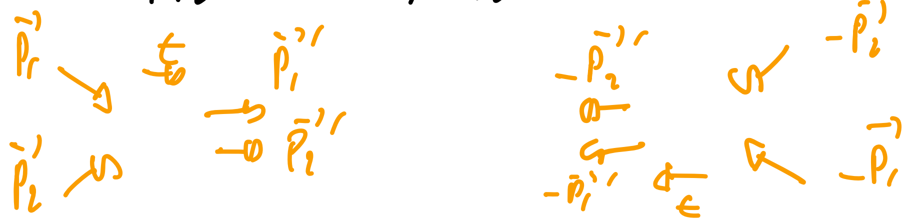
$$dN^+ = dN(\vec p_1{}', \vec p_2{}' \to \vec p_1, \vec p_2)$$
$$= d^3\vec p_2{}'\, d^3\vec p_1{}'\, d^2\sigma(\vec p_1{}', \vec p_2{}' \to \vec p_1, \vec p_2)\; \hat f_1(\vec p_1{}', \vec q_1)\, \hat f_1(\vec p_2{}', q_1)\, |\vec v_1{}' - \vec v_2{}'|\, \tau$$

* Time reversal symmetry,

$\vec p_1{}', \vec p_2{}' \to \vec p_1, \vec p_2$ & $-\vec p_1, -\vec p_2 \to -\vec p_1{}', -\vec p_2{}'$ are as efficient

* Parity symmetry rotate everything by π in $\vec p_1{}', \vec p_2{}'$ plane

$\vec p_1{}', \vec p_2{}' \to \vec p_1, \vec p_2$ & $-\vec p_1{}', -\vec p_2{}' \to -\vec p_1, -\vec p_2$ are as efficient



⇒ Forward & backward scattering are as efficient

$$\Rightarrow \boxed{d^2\sigma(\vec p_1{}', \vec p_2{}' \to \vec p_1, \vec p_2) = d^2\sigma(\vec p_1, \vec p_2 \to \vec p_1{}', \vec p_2{}')}$$

* Collision = rotation of $\vec p_d = \dfrac{\vec p_1 - \vec p_2}{2}$

$$\to |\vec v_1{}' - \vec v_2{}'| = |\vec v_1 - \vec v_2|$$

$$K = \begin{vmatrix} \dfrac{d\vec p_1}{d\vec p_{cm}} & \dfrac{d\vec p_1}{d\vec p_d} \\ \dfrac{d\vec p_2}{d\vec p_{cm}} & \dfrac{d\vec p_2}{d\vec p_d} \end{vmatrix} = 2^3 = 8$$

$$\to \vec p_1 = \vec p_{cm} + \vec p_d \; ; \; \vec p_2 = \vec p_{cm} - \vec p_d$$

$$\to \boxed{d\vec p_1 d\vec p_2 = K\, d\vec p_{cm} d\vec p_d = K\, d\vec p_{cm}{}'\, d\vec p_d{}' = d\vec p_1{}' d\vec p_2{}'}$$

$\vec p_{cm} = \vec p_{cm}{}'$

$\vec p_d = Rot(\theta, \varphi)\, \vec p_d{}'$

Explicit: $\vec{p}_1{}' = \frac{1}{2}(\vec{p}_1 + \vec{p}_2) + R_\Omega \cdot \frac{1}{2}(\vec{p}_1 - \vec{p}_2) = \frac{1}{2}[Id + R_\Omega]\cdot\vec{p}_1 + \frac{1}{2}[Id - R_\Omega]\cdot\vec{p}_2$

$$\vec{p}_2{}' = \frac{1}{2}(\vec{p}_1 + \vec{p}_2) - R_\Omega \cdot \frac{1}{2}(\vec{p}_1 - \vec{p}_2) = \frac{1}{2}[Id - R_\Omega]\cdot\vec{p}_1 + \frac{1}{2}[Id + R_\Omega]\cdot\vec{p}_2 \qquad (3)$$

$$\text{Jacobian} = \frac{1}{2^6}\begin{array}{c} \begin{vmatrix} Id + R_\Omega & Id - R_\Omega \\ Id - R_\Omega & Id + R_\Omega \end{vmatrix} \\ c_1 \qquad c_2 \end{array} = \frac{1}{64}\begin{array}{cc} l_1 & \\ l_2 & \end{array}\begin{array}{c} \begin{vmatrix} Id + R_\Omega & 2\,Id \\ Id - R_\Omega & 2\,Id \end{vmatrix} \\ |c_1 \;\; c_1 + c_2| \end{array} = \frac{1}{64}\begin{array}{c} \begin{vmatrix} 2R_\Omega & 0 \\ Id - R_\Omega & 2\,Id \end{vmatrix} \\ \left|\begin{smallmatrix} l_1 - l_2 \\ l_2 \end{smallmatrix}\right| \end{array} = \frac{64}{64}$$

$$\boxed{\Rightarrow dN^+ = d^3\vec{p}_2\, d^3\vec{p}_1\, d^2\sigma(\theta,\varphi)\,|\vec{v}_1 - \vec{v}_2|\,\tau\, \hat{f}_1(\vec{p}_1{}', \vec{q}_1, t)\,\hat{f}_1(\vec{p}_2{}', \vec{p}_2, t)}$$

All in all, from:

$$\hat{f}_1(\vec{q}_1, \vec{p}_1, t+\tau)V_c|d^3\vec{p}_1| - \hat{f}_1(\vec{q}_1, \vec{p}_1, t)V_c|d^3p| \underset{col}{=} N^+(\tau) - N^-(\tau)$$

we get
$$\left.\frac{\partial \hat{f}_1}{\partial t}\right|_{col} \simeq \frac{N^+(\tau) - N^-(\tau)}{V_c\,|d^3p|\,\tau}$$
$$= \int d^3\vec{p}_2\, d^2\sigma\, |\vec{v}_1 - \vec{v}_2| \left[\hat{f}_1(\vec{q}_1, \vec{p}_1{}')\hat{f}_1(\vec{q}_1, \vec{p}_2{}') - \hat{f}_1(\vec{q}_1, \vec{p}_1)\hat{f}_1(\vec{q}_1, \vec{p}_2)\right]$$

This leads to the celebrated Boltzmann equation (BE)

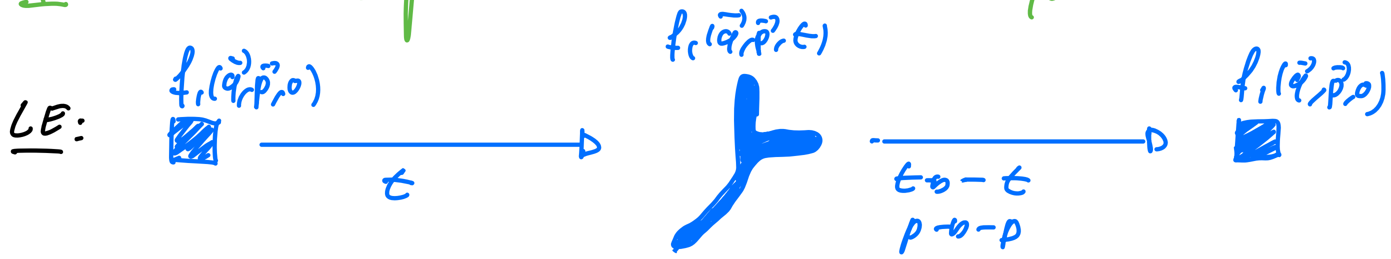
$$\boxed{\frac{\partial \hat{f}_1(\vec{q}_1, \vec{p}_1, t)}{\partial t} + \{\hat{f}_1, H_1\} = \int d^3\vec{p}_2\, d^2\sigma\, |\vec{v}_1 - \vec{v}_2| \left[\hat{f}_1(\vec{q}_1, \vec{p}_1{}')\hat{f}_1(\vec{q}_1, \vec{p}_2{}') - \hat{f}_1(\vec{q}_1, \vec{p}_1)\hat{f}_1(\vec{q}_1, \vec{p}_2)\right]}$$

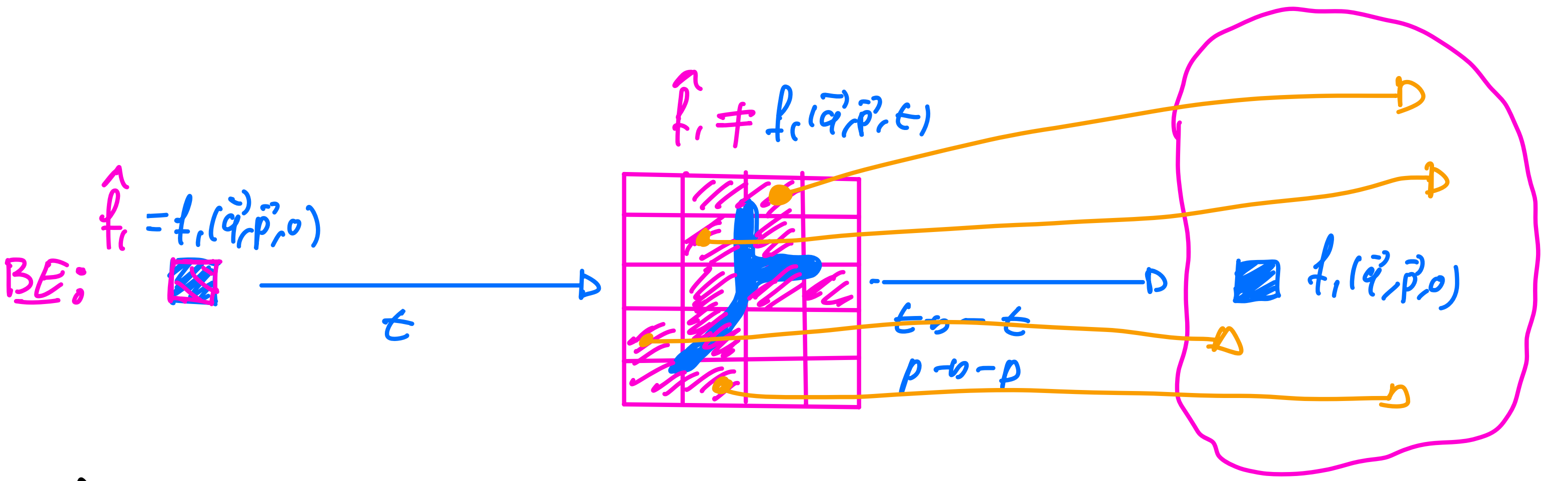
$\Rightarrow$ closed evolution equation for $\hat{f}_1(\vec{q}, \vec{p}, t)$ (BE)

2.3) The H theorem & the convergence to equilibrium

Let us show that if $\hat{f}_1$ solves the B. Eq., then it relaxes irreversibly towards $\hat{f}_1^{eq} = \frac{N}{Z} e^{-\beta H_1}$

Q: How is this possible since Liouville's equation is reversible?





The fractions of initial conditions compatible with coming back is $< 1 \Rightarrow$ irreversibility. How do we know that?

$\rightarrow$ Boltzmann H-theorem.

The H theorem: Let f be a solution of the Boltzmann Eq°, then

$$H(t) = \int d\vec p\, d\vec q\; f(\vec q,\vec p,t)\ln f(\vec q,\vec p,t)$$

is a decreasing function of t.

Notation: For clarity, we drop the hat $\hat f_1 \to f_1$ & the "1", $f_1 \to f$. We also often write $f(\vec p)$ as a proxy for $f(\vec q,\vec p,t)$.

f_1 & f_2 then refers to $f(\vec q,\vec p_1)$ & $f(\vec q,\vec p_2)$.

f_1' & f_2' ——— $f(\vec q,\vec p_1')$ & $f(\vec q,\vec p_2')$

Proof: $\frac{d}{dt} H(t) = \int d\vec{p}\, d\vec{q}\ \partial_t \left[f \ln f \right] = \int d\vec{p} d\vec{q} \left(\ln f + 1 \right) \partial_t f$ (5)

$$= \int \underbrace{d\vec{p} d\vec{q}}_{= d\Gamma} \ln f \cdot \partial_t f + \underbrace{\partial_t \underbrace{\int d\vec{p} d\vec{q}\, f}_{N}}_{=0}$$

Let us now use the BE to replace $\partial_t f$:

$$\frac{d}{dt} H(t) = \underbrace{\int d\Gamma \ln f \left[\frac{\partial H}{\partial \vec{q}} \cdot \underbrace{\frac{\partial f}{\partial \vec{p}}}_{\text{IBP}} - \frac{\partial H}{\partial \vec{p}} \cdot \underbrace{\frac{\partial f}{\partial \vec{q}}}_{\text{IBP}} \right]}_{1} + \underbrace{\int d\vec{p}_1 d\vec{q}\, d\vec{p}_2\, d\sigma\, |\vec{v}_1 - \vec{v}_2| \left(f_1' f_2' - f_1 f_2 \right) \ln f_1}_{2}$$

$$1 = \int d\Gamma \left[-\frac{1}{\cancel{f}} \frac{\partial f}{\partial \vec{p}} \cdot \frac{\partial H}{\partial \vec{q}} \cancel{f} + \frac{1}{\cancel{f}} \frac{\partial f}{\partial \vec{q}} \cdot \frac{\partial H}{\partial \vec{p}} \cancel{f} \right] = \int d\Gamma\, -\frac{\partial}{\partial \vec{p}} \cdot \left(f \frac{\partial H}{\partial \vec{q}} \right) + \frac{\partial}{\partial \vec{q}} \cdot \left(f \frac{\partial H}{\partial \vec{p}} \right)$$

$$= 0$$

In 2, $\vec{p}_1$ & $\vec{p}_2$ are dummy variables so

$$(2 \text{ with } \vec{p}_1, \vec{p}_2) = \frac{1}{2} (2 \text{ with } \vec{p}_1, \vec{p}_2) + \frac{1}{2} (2 \text{ with } \vec{p}_2, \vec{p}_1)$$

$$2_a = \frac{1}{2} \int d\vec{p}_1\, d\vec{q}\, d\vec{p}_2\, \underbrace{d\sigma(\vec{p}_1, \vec{p}_2 \to \vec{p}_1{}', \vec{p}_2{}')}\ \underbrace{|\vec{v}_1 - \vec{v}_2|} \left(f_1' f_2' - f_1 f_2 \right) \left[\ln f_1 + \ln f_2 \right]$$

$$= d\sigma(\vec{p}_1{}', \vec{p}_2{}' \to \vec{p}_1, \vec{p}_2)\ |\vec{v}_1{}' - \vec{v}_2{}'|\ \left(f_1' f_2' - f_1 f_2 \right) \left[\ln f_1 + \ln f_2 \right]$$

& $d\vec{p}_1 d\vec{p}_2 = d\vec{p}_1{}' d\vec{p}_2{}'$

$\vec{p}_1$ & $\vec{p}_2$ are the images of $\vec{p}_1{}'$ & $\vec{p}_2{}'$ by $\Rightarrow$ $\vec{p}_1 = (\vec{p}_1{}')'$

$\vec{p}_2 = (\vec{p}_2{}')'$